PROJECT DESCRIPTION: THE GEOMETRY AND TOPOLOGY OF HEEGAARD SPLITTINGS

1. Introduction

Now that the geometrization conjecture, the tameness conjecture and the ending lamination conjecture have been proved, few of the major open problems that drove the last thirty years of 3-manifold topology remain. However, the geometry and topology of 3-manifolds is still far from well understood. There remain many open questions about surfaces in 3-manifolds, combinatorial structures on 3-manifolds and how these relate to the geometry of a 3-manifold. This is the beginning of a new phase in the field in which a much stronger direct connection is forged between geometry and topology. The research program I propose below is fundamental to this new direction.

A *Heegaard splitting* is a decomposition of a closed 3-manifold into two handlebodies (regular neighborhoods of graphs) that coincide along their boundaries. This common boundary is called a *Heegaard surface*. Since soon after Poincare initiated the study of 3-manifolds in 1902, Heegaard splittings have been a central tool in the field, providing examples and counter examples, including the first construction of a homology sphere. Today they are the basis for Ozsvath and Szabo's Heegaard floer homology, Lackenby's virtually Haken conjecture program and an active field studying Heegaard splittings in and of themselves. There have also been a number of recent results relating topological properties of a Heegaard splitting to properties of the hyperbolic structure on the ambient 3-manifold. All together, these directions make Heegaard splittings a vitally important part of modern 3-dimensional topology.

The purpose of this project is to further understand the isotopy classes of Heegaard splittings in any given 3-manifold. This includes understanding which manifolds admit multiple non-isotopic Heegaard splittings of the same genus, how these distinct Heegaard splittings are related to each other, and the symmetries (i.e. mapping class groups) of these Heegaard splittings. These problems have historically proved difficult because Heegaard surfaces are highly compressible, so any approach based on the fundamental group of the 3-manifold must either be very subtle or is doomed to failure. Recently developed geometric techniques have proved much more robust. In addition to developing new techniques, we propose to further explore two approaches that the investigator has had particular success with. The first is the double sweep-out method that the investigator used to topologically interpret and generalize a complex geometric construction by Hass, Thompson and Thurston [15]. The second is an expanded approach to thin position, furthering ideas of Rubinstein and Bachman, that the investigator has used to solve the isotopy problem for Heegaard splittings of hyperbolic knot complements.

A series of open problems that are fundamental to understanding Heegaard splittings, and on which we propose to work are described in Section 2. In Sections 3 and 4, we describe the two approaches mentioned above - double sweep-outs and thin position - and discuss how we hope to apply them to the open problems. In Section 5, we describe a third, somewhat speculative, approach that we believe will prove even more effective in understanding these problems - applying thin position arguments to the Rubinstein-Scharlemann graphic.

Finally, we propose to work on two additional projects that are closely related to the research described so far. In section 6 we describe a project with Daryl McCullough studying the placement space of a Heegaard surface, a generalization of the placement space for a knot defined by Hatcher [16]. In Section 7 we describe a project with Yair Minsky and Ian Biringer attempting to understand when a pseudo-Anosov automorphism of the boundary of a handlbody will extend to the entire handlebody, based on the stable lamination of the automorphism.

The intended broader impacts from this project are described in Section 8 and the results from the investigator's NSF postdoctoral grant are described in Section 9.

2. Background

2.1. **Distinguishing Heegaard splittings.** The problem of classifying the Heegaard splittings of a given 3-manifold can be broken down into two parts: First one must characterize the Heegaard splittings, i.e. find a list of surfaces such that any Heegaard surface is isotopic to one on the list. Second, one must determine which of the surfaces on this list are isotopic to each other.

Most of the simple 3-manifolds that one tends to think of have a unique Heegaard splitting for any given genus. This is true for the 3-sphere [46], lens spaces [8], the 3-torus [6] and for any surface product [43]. Manifolds with distinct isotopy classes of Heegaard splittings, on the other hand, require some fairly non-trivial topology. These have been constructed by Lustig and Moriah [29], [30], Kobayashi [26], Morimoto and Sakuma [34], Bachman and Derby-Talbot [5], Bachman [3] and by the investigator [21]. (The examples in [30] and [26] are based on unpublished work of Casson and Gordon.)

The problem of distinguishing Heegaard splittings has seen a fair amount of progress, but while there are trends in the intuition between these different examples, the methods of construction and the methods of distinguishing the Heegaard splittings are very different. Thus we propose the following problem:

Problem 1. *Find a unified approach to distinguishing Heegaard splittings.*

Meanwhile, the problem of characterizing the Heegaard splittings of a given 3-manifold (the first half of classification) has not been nearly as successful. It has been carried out for Seifert fibered spaces [33], two-bridge knot complements [27] and torus bundles [12], but the list ends there. Scharlemann and Tomova's theorem [42] allows one to construct hyperbolic 3-manifolds in which there are unique Heegaard splittings below a given genus, but it does not allow one to rule out unexpected Heegaard splittings above that threshold. In particular, there are no closed hyperbolic 3-manifolds whose Heegaard splittings have been characterized (let alone classified). This suggests our next problem:

Problem 2. *Characterize all Heegaard splittings of some hyperbolic 3-manifold.*

While this may sound like a very modest problem, it is likely that any methods that work on a single hyperbolic 3-manifold will generalize to a much larger class and will lend insight into classifying Heegaard splittings in general. Approaches to both of these problems will be described below.

2.2. **Common Stabilizations.** A *stabilization* of a Heegaard splitting is the result of attaching one or more trivial handles to each handlebody, producing a new Heegaard splitting (for the same 3-manifold) with higher genus. This construction is equivalent to taking the connect sum with a Heegaard splitting of S^3. Reidemeister [37] and Singer [44] proved independently in 1933 that given a 3-manifold M with Heegaard surfaces Σ and Σ', there is a third Heegaard surface Σ'', called a *common stabilization*, that is isotopic to a stabilization of Σ and to a stabilization of Σ'. The *stable genus* of Σ and Σ' is the genus of their smallest common stabilization.

Given positive integers p and q, one can ask what are the possible stable genera of all pairs of Heegaard splittings with genera p and q. For the rest of this proposal, we will let $s(p,q)$ be the smallest function such that any two genus p and $q \leq p$ Heegaard splittings of the same 3-manifold have stable genus less than or equal to $s(p,q)$. Rubinstein and Scharlemann [38] proved that for non-Haken 3-manifolds, $s(p,q) \leq 5p+8q-9$ and used similar techniques to find a quadratic bound for Haken 3-manifolds [39]. It is widely believed that $s(p,q)$ should be much lower, but no better bound solely in terms of p and q has been proved. (The investigator showed that the stable genus can be bounded in terms of the number of inflection points and cusps in the Rubinstein-Scharlemann graphic for two sweep-outs [24].

A lower bound on $s(p,q)$ has proved almost as difficult to find. Because there are 3-manifolds with distinct Heegaard splittings, $s(p,q)$ must be at least $p+1$. The first evidence that $s(p,q)$ is greater than $p+1$ did not appear until the end of 2007. At the AIM workshop on Heegaard splittings, Triangulations and Hyperbolic Geometry, two results were announced: First, Dave Bachman announced examples showing that $s(2k, 2k-1) \geq 3k-3$. (These examples did not appear in print until much later.) Second, Joel Hass and Abby Thompson announced joint work with Bill Thurston using completely different techniques to show that there are Heegaard splittings for which the smallest stabilization whose handlbodies can be interchanged by an isotopy of the 3-manifold has genus twice the original genus. In other words, the original Heegaard splitting and the Heegaard splitting with its handlebodies switched have stable genus twice its original genus; this is an oriented version of the stabilization problem. This second proof also gives important insight into the unoriented problem.

In the following months, I found a combinatorial version of Hass, Thompson and Thurston's proof [22], showing that the number of stabilizations needed to interchange the handlebodies of a Heegaard splitting is bounded below by the smaller of half the Hempel distance or twice the genus of the original splitting. I then generalized these methods in [21] to address the stabilization problem directly, showing that for every positive integer k, there is a 3-manifold with Heegaard splittings of genus $2k$ and $2k-1$ whose smallest common stabilization has genus $3k-1$. Thus the function $s(p,q)$ is at least $p + \frac{1}{2}(q-1)$. Since this preprint appeared, a Japanese graduate student has improved the proof to show $s(p,q) \geq p + \frac{1}{2}q$ [45]. This paper does not yet appear to be publicly available.

The difference between the known upper bound of $5p + 8q - 9$ and lower bound of $p + \frac{1}{2}q$ on $s(p,q)$ is still quite large. It is likely that the actual value is much closer to the lower bound than the upper bound. While the value of $s(p,q)$ is not as interesting as the examples, it provides important motivation for asking more subtle questions. Thus one of the main objectives remains understanding $s(p,q)$.

Problem 3. *Find sharp bounds on the stable genera of pairs of Heegaard splittings, i.e. determine the exact value of* $s(p,q)$.

For both of the approaches to lower bounds on $s(p,q)$ described above, the examples are constructed in roughly the same way, by gluing together two or more manifolds along their boundaries using a complicated gluing map. In these cases, the genera of p, q, and the stable genus are determined by the genera of the gluing surfaces. It is conceivable that every pair of Heegaard splittings with high stable genus can be constructed in this way, in which case an upper bound on $s(p,q)$ would be almost immediate. However, the existence of other types of examples remains an open problem.

Problem 4. *Are there other constructions that produce pairs of Heegaard splittings with high stable genus?*

2.3. **Mapping class groups.** While considering the stabilization problem, as well as other aspects of Heegaard splittings, one often runs into the problem of understanding the symmetries of a Heegaard splitting. The *mapping class group* of a Heegaard splitting is the group of automorphisms of the ambient 3-manifold that take the Heegaard surface to itself, modulo isotopies that keep the Heegaard surface on itself. The investigator and Hyam Rubinstein constructed a number of examples of Heegaard splittings with non-trivial mapping class groups.

Mapping class groups of Heegaard splittings have been studied in many guises and from many angles. Goeritz [14] found a generating set for the mapping class group of the genus two Heegaard splitting of the 3-sphere. Scharlemann recently found a new proof of this [40], and Akbas [1] and Cho [11] upgraded the generating set to a finite presentation. Two published proofs of generating sets for higher genus splittings of the 3-sphere have since been found faulty and it is currently unknown whether these groups are finitely generated.

Problem 5. *Are mapping class groups of Heegaard splittings of* S^3*, or of other 3-manifold finitely generated?*

Meanwhile, Zimmerman [47] has studied a number of examples of finite mapping class groups of Heegaard splittings of hyperbolic 3-manifolds. Namazi [35] showed that Heegaard splittings with sufficiently high Hempel genus have finite mapping class groups and Birman and Rubinstein classified mapping class groups of certain one sided Heegaard splittings [7]. However, the results in these papers are for the most part disconnected, apparently written without knowledge of each other.

In 2006, Hyam Rubinstein and the investigator set out to create a unified framework for studying mapping class groups of Heegaard splittings. Our results included a characterization of finite subgroups of the mapping class group and a distance bound for Heegaard splittings admitting reducible automorphisms. The following year, the investigator found a finite set of generators for the mapping class group of the genus three Heegaard splitting of the 3-torus [20], and more recently improved Namazi's, showing that every Heegaard splitting with Hempel distance greater than three has a finite mapping class group [23]. Heegaard splittings with Hempel distance two and infinite mapping class groups are known to exist, but it remains unknown whether a distance three Heegaard splitting can have an infinite mapping class group.

By the investigator's work with Hyam Rubinstein, a Heegaard splitting of a hyperbolic 3-manifold will have a finite mapping class group if and only if the mapping class group of the Heegaard splitting injects into the mapping class group of the 3-manifold. In particular, finite mapping class groups are induced by isometries of the ambient 3-manifold in this case. Thus the important dichotomy seems to be between finite mapping class groups, which are geometric, and infinite mapping class groups which are not.

Problem 6. *Characterize Heegaard splittings with infinite mapping class groups topologically. In particular, is there a Heegaard splitting with Hempel distance three and an infinite mapping class group?*

There is still relatively little known about mapping class groups of Heegaard splittings and there are many more open questions that have barely been explored. One question which appears to be extremely difficult, and which is related to finite generation is whether mapping class groups of Heegaard splittings are homologically stable.

A sequence of groups is said to be homologically stable if for any dimension n, the sequence of n-dimensional homology groups is eventually constant. Mapping class groups of general surfaces are known to be homologically stable as the genus goes to infinity. Given a Heegaard splitting, one can stabilize it repeatedly and consider the mapping class groups of these splittings. The mapping class groups are subgroups of the mapping class group of the abstract surface, so it is reasonable to ask the same question about them.

Problem 7. *Are mapping class groups of Heegaard splittings homologically stable?*

3. Double sweep-outs

The double sweep-out or graphic method was developed by Rubinstein and Scharlemann [38] as a way of dealing with the fact that Heegaard splittings are highly compressible. When comparing a pair of Heegaard splittings to each other, one runs into the problem that they may intersect each other in a variety of different ways, most of which are topologically insignificant. The double sweep-out replaces each individual surface with a continuous family of surfaces (a sweep-out) parameterized by an interval. This replaces a single pair of surfaces with a square of pairs, parameterized by the two variables.

Rubinstein and Scharlemann's approach cuts the square into regions along a graph defined by pairs of surfaces that are tangent. They then analyze these regions in order to find one in which the pair of surfaces intersect in a meaningful way, i.e. a way that is useful for the rest of their argument.

While this approach has proved very useful, there is a great deal more information that can be culled from the graphic. Insight into such an approach can be found from a geometric point of view. Hass, Thompson and Thurston's work [15] on stabilizations of Heegaard splittings uses a pair of geometric sweep-outs in which one sweep-out determines a pinched hyperbolic structure on the 3-manifold and the other consists of (immersed) harmonic surfaces. In particular, the pinched hyperbolic geometry comes from taking the infinite cyclic cover of a hyperbolic surface bundle, cutting out a compact portion of it, then capping it off with handlebodies, using a method due to Namazi and Souto [36].

This construction gives a very precise geometric picture of the 3-manifold, as a long thin product region with caps. Because the second sweep-out is harmonic, they bound the genus of the surfaces based on how they divide the volume of the product region. The picture they derive from this is fairly nice. They show that the harmonic sweep-out either faces one direction or the other with respect to the other sweep-out, in a well defined sense, or it cuts it transversely, in which case it must have high genus. They use this to find a lower bound on the number of stabilizations needed to "flip" the harmonic sweep-out from one way to the other.

Hass, Thompson and Thurston's proof involves many technicalities for dealing with a pinched negatively curved metric and an immersed (rather than embedded) sweep-out. However, the intuition that comes from it is invaluable and can be interpreted via Rubinstein and Scharlemann's graphic. In [22] the investigator did precisely this, producing a completely topological proof of the same result. In particular, the length of the product region is replaced by the Hempel distance (i.e. the curve complex distance) of the corresponding Heegaard splitting.

In a sequel [21], the investigator generalized their result further, using the flexibility that comes from the topological setting to deal with 3-manifolds that are built by gluing together pieces along incompressible surfaces. This again produces a very intuitive picture, based around the idea that for each of the pieces, a Heegaard splitting of the whole 3-manifold must face one way or the other with respect to a sweep-out of that piece, or it must cut through it transversely, in which case the genus of the Heegaard splitting will be bounded in terms of the complexity of that piece (which in this case is a distance in the curve complex).

The topological approach has two benefits: In addition to producing new results about Heegaard splittings, it strengthens the connection between the geometry and topology of Heegaard splittings being explored by Brock, Minsky, Namazi and Souto, as an extension of the original work by Namazi and Souto [36]. For example, their work suggests that if a Heegaard splittings is "highly twisted" along a subsurface then every other Heegaard splitting for the 3-manifold should either be parallel or be forced to have high genus. Yoav Moriah, Yair Minsky and the investigator have shown precisely this by a topological argument generalizing the double sweep-out methods of the investigator [25].

The double sweep-out methods developed so far have proven exceptionally useful for characterizing Heegaard splittings, understanding the connection between the geometry and topology of 3-manifolds, and for simplifying the proofs of earlier results in the field of Heegaard splitting. The ongoing work of Brock, Minsky, Namazi and Souto suggests that the analogy between geometry and topology should continue to grow, leading to new applications of the double sweep-out method, which in turn will further cement the analogy.

In particular, their work suggests that the set of all 3-manifolds whose geometry is bounded in the appropriate way can be constructed by gluing together a finite collection of 3-dimensional pieces along their boundaries. A number of recent results have considered the Heegaard splittings of such 3-manifolds, but we expect that the double sweep-out methods developed by the author, along with this geometric intuition, should lead to much stronger results. We thus suggest the first, rather broad, objective for this project:

Objective 1. *Apply the double sweep-out methods to understand Heegaard splittings of 3-manifolds constructed by gluing 3-dimensional pieces along compressible and incompressible boundary components.*

4. Iterated thin position

Thin position was first defined for knots by David Gabai [13]. The idea was translated into a technique for Heegaard splittings by Scharlemann and Thompson [41], where it has been employed in a number of important applications, particularly by Schultens [43]. Rubinstein has interpreted thin position via minimal surfaces, in a program that, while technically difficult, has proven invaluable as a source of analogy.

Rubinstein's interpretation of thin position goes as follows: If we were to try to isotope a Heegaard surface to a minimal surface (with respect to some fixed metric on the ambient 3-manifold) we run into the problem that the surface might collapse onto one of the handlebodies and degenerate. To avoid this, one might consider a sweep-out of the Heegaard surface and try to isotope this sweep-out so that the areas of the leaves are somehow minimized. The leaves at the beginning and end collapse, so their areas are zero. To get from one to the other, we have a "path" of surfaces in which the areas increase and then decrease. The maximum in the middle is a surface whose area can be reduced in two different ways. If we choose the most "efficient" family of surfaces, we might expect that this maximal surface will be an index-one minimal surface.

To interpret this topologically, we can think of a Heegaard splitting as a sequence of surfaces, starting with a trivial sphere, in which we get from one surface to the next by compressing or attaching tubes to the previous surface. Rather than minimizing the area, we will minimize the genus of the surfaces. (This link between genus and area comes from the fact that a minimal surface has bounded curvature, so its genus is bounded by its area.) Such a sequence of surfaces is "thin" if the genera of the intermediate surfaces are minimized in the appropriate sense.

Given such a sequence with a single maximal surface, this maximum will be a Heegaard surface for the 3-manifold. Such a sequence is thin if and only if the maximal surface is strongly irreducible, as defined by Casson and Gordon [10]. If there are more than one maxima in the sequence then the maximal and minimal surface define what's called a generalized Heegaard splitting. If such a sequence is thin then the minima will be incompressible surfaces.

This definition of thin position can be generalized further in a way that incorporates the original notion of thin position for knots. Given a link in the 3-manifold, we can consider a sequence of surfaces as before, but require that consecutive surfaces are related by either tubing/compressing in the complement of the link, or isotoping the surface so that it passes through a single tangency with the link. This notion of thin position was defined and explored by Hayashi and Shimokawa [17].

One can expand Rubinstein's analogy with minimal surfaces by considering an isotopy between two sweep-outs whose areas have been minimized. In other words, we consider a smooth family of sweep-outs such that the beginning and ending sweep-outs are minimal. We would like to choose this family so that the areas of the surfaces in the intermediate sweep-outs are minimized as well. Each sweep-out in the family will have a local maximum, and the areas of these local maxima will increase and then decrease. The local maximum among the local maxima can be isotoped to decrease the area in four different directions - two defined by the sweep-out containing it, and two defined by the family of local maxima. Thus we might expect the local maximum among the local maxima to be an index-two minimal surface. This index two minimal surface characterizes the isotopy between the original two index-one surfaces.

Bachman [4] used this idea as the intuition behind his definition of "topological index". With this defintion, he shows that the common stabilization of two strongly irreducible (i.e. index one) Heegaard surfaces has topological index two. He shows this by defining thin position for a sequence of generalized Heegaard splittings, i.e. by iterating the notion of thin position. This is the basis for Bachman's proof of the Gordon conjecture [2].

The investigator has recently introduced two innovations to iterated thin position [19]. Because the definitions become very complicated when iterated, the first innovation is a more precise definition, based on a simplicial complex in which vertices represent embedded surfaces and edges connect surfaces that are related by compression/tubing. This complex satisfies a family of simple axioms from which the main results of thin position follow. This leads to a much clearer picture of iterated thin position and Bachman's work. It has taken the topology community a number of years to digest and accept his proof of the Gordon conjecture, for example, but the axiomatic leads to a much simpler exposition of the proof.

The second innovation is a generalization of the the iterative method to Hayashi and Shimokawa's version of thin position based on this axiomatic approach. The original version of thin position for Heegaard splittings is rather coarse because it replaces area with genus. In order to define a more refined picture, one can introduce into the 3-manifold a link or graph that is somehow topologically significant. Examples of this include the critical fibers of a Seifert fibered space or the 1-skeleton of a nice triangulation. This makes the topological equivalent of "area" finer; rather than just looking at the genus of a surface, one considers the number of intersections with the graph.

By defining thin position with respect to these objects, one can classify the surfaces with non-trivial index in many situations. For example, in the case of a triangulation, the index-n surfaces in the thin position picture are precisely the index-n normal surfaces with respect to the triangulation. By classifying the high index surfaces in this context, then "forgetting" the graph or link, one can characterize the Heegaard splittings of the original 3-manifold. In particular, given a triangulation in which there are finitely many such surfaces (such as one of Lackenby's taught triangulations [28]), this yields an algorithm for calculating the isotopy classes of Heegaard splittings.

This approach has the potential to produce a fairly unified approach to classifying Heegaard splittings; given a 3-manifold, one must find a link or graph that characterizes its topology. If one can characterize the topologically minimal surfaces with respect to this graph, then one gets a classification of Heegaard splittings of the 3-manifold.

Hyperbolic geometry suggests many such objects to try. For example, Minsky's model hyperbolic 3-manifolds suggests a link in the 3-manifold defined by a path in the pants complex for the surface defining the end [31]. Similarly defined loops appear in Brock's work relating the volume of the compact core of a hyperbolic surface bundle to the Teichmuller distance between its ends [9]. For a surface bundle, one can define a similar link in the 3-manifold and it seems likely that the surfaces with non-trivial index with respect to such a link could be classified. This brings up the second objective for this project:

Objective 2. *Find more examples of links and graphs in 3-manifolds with respect to which topologically minimal surfaces can be characterized.*

5. Generalizations of thin position

The axiomatic approach to thin position, in addition to providing a clearer understanding of existing notions of thin position, also suggests ways that thin position might be applied in other contexts. Axiomatic thin position is defined entirely in terms of the combinatorics of the simplicial complex. The topological index, for example, is defined by the homotopy type of the link of a vertex. Thus any context in which a complex satisfying the appropriate axioms, or similar axioms, can be constructed lends itself to thin position.

Two immediate contexts in which the investigator would like to apply the notion of thin position are as follows: The first is thin position with respect to a handlebody or a pair of handlebodies. This is very similar to thin position with respect to a graph that is the spine of a handlebody, except that one is not forced to choose a set of meridians for the handlebody. This definition will recover stronger versions of many of the results that come from studying double sweep-outs.

The second type of thin position the investigator would like to study is thin position with respect to a sweep-out. This is a slightly more subtle and complex notion than thin position with respect to a handlebody because in addition to measuring how a surface intersects the spine of the handlebody, one can understand precisely how it moves through the product region. The end product would be a method of simplifying the Rubinstein-Scharlemann graphic. It should also lead to a very precise understanding of how two Heegaard splittings of the same 3-manifold are related.

There are probably even more imaginative applications of thin position outside the realm of 3-manifold topology. A theory of thin position generally corresponds to a family of Morse functions and thus can be defined on any space where there is a Morse theory. However, the idea of minimizing the maxima of a presentation is much more general. Thus we propose to work on the following:

Objective 3. *Find more areas in which thin position can be fruitfully applied.*

6. Additional Project: The placement space of a Heegaard surface

Consider a surface S embedded in a 3-manifold M. Let $Emb(M,S)$ be the set of all re-embeddings of S into M such that the re-embedded surface is isotopic to S. The *placement space* $Pl(M,S)$ of S is the quotient of $Emb(M,S)$ by the set of homeomorphisms from S to itself. In other words, the placement space ignores any markings on S and only considers the set defined by the image of the embedding. Any isotopy of S determines a path in $Pl(M,S)$, and any isotopy that returns S to itself determines a loop in $Pl(M,S)$.

If we consider the mapping class group $Mod(M,\Sigma)$ of a Heegaard surface Σ, any element in the kernel of the inclusion map $Mod(M,\Sigma) \to Mod(M)$ defines an isotopy of Σ that returns Σ to itself, and thus defines a loop in $Pl(M,\Sigma)$. The kernel is called the *isotopy subgroup* of $Mod(M,\Sigma)$. There is clearly a strong relation between the fundamental group of the placement space and the isotopy subgroup of the mapping class group. Daryll McCullough and the investigator have been working on using techniques developed by Daryll in other contexts to prove the following conjecture:

Conjecture 1. *If M is a hyperbolic 3-manifold then the placement space of any Heegaard surface for M is a classifying space for the isotopy subgroup of its mapping class group.*

This is part of a more general program to understand how the placement space of a Heegaard surface is related to the isotopy subgroup and to the space of self-homeomorphisms of the 3-manifold.

7. Additional Project: Extension of automorphisms to handlebodies

If Σ is the boundary of a handlebody H and $\phi : \Sigma \to \Sigma$ is an automorphism of the surface, one can ask when ϕ is the restriction to Σ of an automorphism of H, or in other words whether one can extend ϕ to an automorphism of the entire handlebody. If ϕ is known to extend to H then it sends the boundary of any properly embedded disk $D \subset H$ to the boundary of another disk in H. Iterating the map ϕ determines a sequence of disks in H. If ϕ is pseudo-Anosov then the boundaries of these disks limit to the stable lamination for ϕ. Thus in the space of projective measured laminations on Σ, the stable lamination of ϕ is in the closure of the set of loops bounding disks in H. The same is true if ϕ extends to a compression body contained in H, but not necessarily to the entire handlebody.

Given a pseudo-Anosov automorphism of a surface Σ bounding a handlbody H, it is known that if the stable lamination of ϕ is not in the closure of the disk set then iterating ϕ takes any loop in Σ arbitrarily far in the curve complex from the set of loops bounding disks. This (or rather a slightly stronger statement) is the bases for Hempel's construction of Heegaard splittings of arbitrarily high distance [18]. It has also been used by Minsky-Moriah-Schleimer [32] to find knots in the 3-sphere whose complements have Heegaard splittings of arbitrarily high distance, answering a long standing question about tunnel numbers and higher genus bridge numbers of knots in S^3.

It is natural to ask about the converse of the above result: If the stable lamination of ϕ is in the closure of the set of loops bounding disks in H, does it necessarily extend to a non-trivial compression body contained in H? Yair Minsky, Ian Biringer and the investigator have found a proof (in preparation) that under these circimstances, some power of ϕ does extend to a non-trivial compression body in H. This is the first step in a larger program to determine how the stable lamination of a pseudo-Anosov relates to its extensions into H. The following are problems that we would like to consider in the future.

Problem 8. *Find a pseudo-Anosov automorphism of Σ that extends to a compression body in H, but not to all of H. Can an automorphism extend to two distinct compression bodies in H?*

Problem 9. *Find a pseudo-Anosov automorphism ϕ of Σ such that ϕ does not extend to H, but some powere of ϕ does.*

Problem 10. *Find conditions on the stable lamination of ϕ that guarantee it will extend to all of H.*

8. Broader Impacts

Since their introduction in the early 1900s, Heegaard splittings have been a vital tool for placing 3-manifolds in an accessible context. They provide a good introduction to geometric topology and an active area of research for young mathematicians. Right now, the core of the theory of Heegaard splittings is appropriate for beginning graduate students. However, new research continues to provide simpler proofs of the main theorems and more intuitive approaches to the fundamental concepts, so that parts of the field are becoming accessible to advanced undergraduates. The research project described here will make it more accessible and eventually lead to problems that are appropriate for an undergraduate thesis or even an REU, which I would be interested in supervising. This will provide a gateway for the students into other areas of algebraic and geometric topology.

In the mean time, I have been working to make other traditionally graduate level subjects accessible to undergraduates. During the summer of 2008 I supervised six Yale undergraduates on a research project studying normal loops in triangulated surfaces. A normal loop is a simple closed curve that intersects the triangles of the triangulation in arcs such that each arc has its endpoints in distinct edges of the triangle. This is a analogous to the study of normal surfaces in 3-manifolds (which has many connections to Heegaard splittings), but it is also closely related to train tracks on a surface.

This second connection suggests that normal loops could be used to investigate aspects of surface topology that train tracks are usually used for: laminations, Teichmüller space, the mapping class group, etc. Moreover, the space of normal loops has many advantages over train tracks, in particular that all normal loops are described by a single, easily defined vector space. During the eight week program, the six students made very exciting progress in reformulating the existing methods of surface topology in terms of normal loops. I hope in the future to continue this research with other groups of undergraduates.

I also created the blog *Low dimensional topology* (*http://ldtopology.wordpress.com/*) in November of 2007 as a way of publicizing recent progress in topology for the mathematical community as a whole. I have mostly discussed the work of other topolgists, but I also included a series of posts on my definition of axiomatic thin position in the hope that others would be interested in finding other contexts for applying thin position. As a way of broadening the scope of the blog, I have recently managed to recruit a number of other topologists to contribute, including Nathan Dunfield.

I have received a great deal of positive feedback from young topologists who have found the blog useful. It has also proved to be a powerful tool for quickly disseminating news about recent advances. Most notably, when Kahn and Markovich announced a claimed proof that every hyperbolic 3-manifold contains an immersed π_1-injective surface, an announcement on the blog began a discussion that ended with a fairly detailed outline, made publicly available. I plan in the future to continue to expand the blog to a wider scope of research and a wider audience, so that it will continue to promote discussion within the research community.

It should be noted that OSU has a substantial population of Native American and other underserved minorities, and Oklahoma is geographically isolated from the academic centers of the country. Through involvement in the PI's research, mathematically talented students at OSU will have the opportunity to develop their talents, increase their visibility and confidence and prepare themselves for further success in mathematics and science.

9. Prior NSF Support

The investigator was funded by NSF MSPRF grant 0602368, from June 2006 to May 2009. During this period, I developed the ideas that I had begun in graduate school into powerful techniques that are effective for solving many of the problems that motivated my initial research. The relatively low teaching load afforded by this grant allowed me to write up this results in a timely manner, and all but the most recent are now available on the mathematics arXiv.

I spent a portion of the first year of this grant visiting the University of Melbourne, where I collaborated with Hyam Rubinstein. We wrote the paper *Mapping class groups of Heegaard splittings* (arXiv:math/0701119), which presents a number of examples of Heegaard splittings with non-trivial mapping class groups, and proves an extension of the Nielsen realization theorem to Heegaard splittings.

After returning from Melbourne, I finished two papers that began as projects in graduate school. The first, *Stable functions and common stabilizations of Heegaard splittings* (arXiv:0705.3712), shows that a bound for the stable genus of two Heegaard splittings can be read from the inflection points and cusps in the Rubinstein-Scharlemann graphic for the two splittings. This paper has been published in Transactions of the AMS. The paper *Generalized handlebody sets and non-Haken 3-manifolds* is a collaboration with Terk Patel, who was a fellow graduate student at UC Davis. We considered the set of loops in the curve complex that represent trivial elements of the homology of the handlebody, but non-trivial elements in the surface, and show that this set is 2-dense in the curve complex. This paper has been published in the Pacific Journal of Mathematics.

I then wrote two papers that study isotopies of Heegaard splittings by replacing them with Morse functions or sweep-outs. In the paper *Automorphisms of the three-torus preserving a genus three Heegaard splitting* (arXiv:0708.2683), I found a finite generating set for the mapping class group of a genus three Heegaard splitting of the 3-torus. The method expands on the analysis of Rubinstein-Scharlemann graphics I had begun earlier. I used a similar idea in *Horizontal Heegaard splittings of Seifert fibered spaces* (arXiv:0802.0856) to prove that a strongly irreducible Heegaard splitting of a Seifert fibered space is uniquely determined (up to isotopy) by the slopes of intersection with a collection of vertical tori. This proof interprets and strengthens an argument of Bachman and Derby-Talbot [5] by looking at the restriction of a sweep-out representing the Heegaard splitting to a vertical torus.

The final two papers I completed during this period are the work that I am most excited about. In *Flipping and stabilizing Heegaard splittings* (arXiv:0805.4422) and *Bounding the stable genera of Heegaard splittings from below* (arXiv:0807.2866) I found and extended a combinatorial version of the argument in [15] to find examples of pairs of Heegaard splittings whose stable genus is larger than previous examples. The argument, based on the Rubinstein-Scharlemann graphic, is a very natural extension of the analysis of the graphic that I developed earlier in this period.

I spent the final year of this period developing and perfecting the axiomatic approach to thin position described in Section 4. I am very close to finishing the manuscript and hope to have it publicly available by the end of the year. In the mean time, I outlined the program on the blog *Low Dimensional Topology*. During this final year, I also completed two collaborations: *Flipping bridge surfaces* (arXiv:0908.3690) with Maggy Tomova extends to bridge positions for knots the double sweep-out methods that I used to bounded stable genus of Heegaard splittings. The paper *On the existence of high index topologically minimal surfaces* with Dave Bachman, shows that Bachman's topological index theory is non-trivial, i.e. that there are surfaces with arbitrarily high topological index.

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